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Finite p-groups with non-cyclic center have a non-inner automorphism of order p

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Abstract. Let G be a finite non-abelian p-group and let $W(G)/Z(G) = \Omega_1(Z_2(G)/Z(G))$. A longstanding conjecture asserts that G admits a non-inner automorphism of order p. We confirm the conjecture in case $Z(G)$ is not cyclic and $W(G)$ is non-abelian.

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1. Introduction. Let G be a finite non-abelian p-group. In 1973, Berkovich [9, Problem 4.13] posed the following conjecture: Every finite non-abelian p-group admits a non-inner automorphism of order p. The conjecture is confirmed for many classes of finite p-groups. For example, see [1,2,5,7,8,10,11], and [3] for other references. In most of the cases, it is observed that if the conjecture is false for G, then the center $Z(G)$ is cyclic.

In [4, Lemma 4.2], Abdollahi et al. proved the existence of a non-inner automorphism of order p which fixes $\Phi(G)$ element-wise in case of a finite non-abelian p-group G of class 3 with non-cyclic center $Z(G)$, where $p > 2$. In [4, Theorem 5.1(1)], they also confirmed the conjecture for a finite 2-group G of class 3 with non-cyclic center $Z(G)$ and $d(G) \neq 3$. The main motivation behind this paper is the following natural question which arises from [4, Lemma 4.2 and Theorem 5.1(1)]:

Question. *Given a finite non-abelian p-group G with non-cyclic center $Z(G)$, under what conditions does the conjecture hold?*

In this short note, we answer this question. We confirm the conjecture in case $Z(G)$ is not cyclic and $W(G)$ is non-abelian. The main result of this note is the following theorem:

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Theorem 1.1. *Let G be a finite non-abelian p-group with non-abelian $W(G)$ such that every automorphism of G of order p, which fixes $\Phi(G)$ element-wise, is inner. Then $Z(G)$ is cyclic.*

2. Proof of Theorem 1.1. Let $W(G)/Z(G) = \Omega_1(Z_2(G)/Z(G))$. We first prove that $C_G(W(G)) = \Phi(G)$. Let M be a maximal subgroup of G and let $g_0 \in G - M$. Let $z_0 \in Z(G) \cap M$ be of order p. Then the map $\alpha : G \to G$ defined as $\alpha(mg_0^i) = mg_0^i z_0^i$ for all $m \in M$ is easily seen to be an automorphism of order p which fixes $\Phi(G)$ element-wise. By assumption, $\alpha = \theta_{a_M}$, the inner automorphism induced by some $a_M \in G$. It is easy to see that $a_M \in W(G)$ and $M = C_G(a_M)$. Since $[W(G), \Phi(G)] = 1$, $\Phi(G) \leq C_G(W(G))$. It thus follows that

$$\Phi(G) \leq C_G(W(G)) \leq \bigcap_M C_G(a_M) = \bigcap_M M = \Phi(G),$$

and hence $C_G(W(G)) = \Phi(G)$.

It follows from [6, Remark 1] that $Z(G) \leq \Phi(G)$. For $x \in G$ and $H \leq G$, let $\overline{x}$ and $\overline{H}$ denote the coset $x\Phi(G)$ and the quotient group $H\Phi(G)/\Phi(G)$ respectively. For each $w \in W(G)$, the map $f_w : \overline{W(G)} \to \Omega_1(Z(G))$ defined as $f_w(\overline{v}) = [v, w]$ for all $v \in W(G)$ is easily seen to be a homomorphism. The map

$$f : W(G) \to \mathrm{Hom}(\overline{W(G)}, \Omega_1(Z(G)))$$

defined as $f(w) = f_w$ for all $w \in W(G)$ is then a homomorphism with

$$\ker(f) = W(G) \cap C_G(W(G)) = W(G) \cap \Phi(G).$$

We prove that f is an epimorphism. Let $\alpha \in \mathrm{Hom}(\overline{W(G)}, \Omega_1(Z(G)))$. Since $\overline{G}$ is elementary abelian, the homomorphism α can be extended to the homomorphism $\beta : \overline{G} \to \Omega_1(Z(G))$ defined by

$$\beta(\overline{w}) = \begin{cases} \alpha(\overline{w}) & \text{if } \overline{w} \in \overline{W(G)}, \\ 1 & \text{if } \overline{w} \in \overline{G} - \overline{W(G)}. \end{cases}$$

Since $\Omega_1(Z(G)) \leq \Phi(G)$, the map $\sigma_\beta : G \to G$ defined as $\sigma_\beta(g) = g\beta(\overline{g})$ is easily seen to be an automorphism of order p which fixes $\Phi(G)$ element-wise. By hypothesis, $\sigma_\beta = \theta_a$ for some $a \in W(G)$. Therefore, for each $w \in W(G)$,

$$\alpha(\overline{w}) = \beta(\overline{w}) = w^{-1}\sigma_\beta(w) = w^{-1}\theta_a(w) = [w, a] = f_a(\overline{w}),$$

and hence f is an epimorphism. It follows that

$$\frac{W(G)}{W(G) \cap \Phi(G)} \simeq \overline{W(G)} \simeq \mathrm{Hom}(\overline{W(G)}, \Omega_1(Z(G))),$$

and hence $d(Z(G)) = 1$. □

Observe that Theorem 1.1 can be restated in the following way:

Let G be a finite non-abelian p-group such that $W(G)$ is non-abelian and $Z(G)$ is not cyclic. Then there exists a non-inner automorphism of order p which fixes $\Phi(G)$ element-wise.

We conclude the paper by giving an example of a group which supports our result. Consider

$$G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8 \rangle,$$

with relations: $f_1^2 = f_6, f_2^2 = f_3^2 = f_4^2 = f_6^2 = f_8^2 = [f_3, f_1] = [f_4, f_1] = [f_6, f_1] = [f_8, f_1] = [f_3, f_2] = [f_4, f_2] = [f_6, f_2] = [f_8, f_2] = [f_5, f_3] = [f_6, f_3] = [f_7, f_3] = [f_8, f_3] = [f_5, f_4] = [f_6, f_4] = [f_7, f_4] = [f_8, f_4] = [f_6, f_5] = [f_7, f_5] = [f_8, f_5] = [f_7, f_6] = [f_8, f_6] = [f_8, f_7] = 1, [f_2, f_1] = f_5, f_7 = [f_5, f_1] = [f_5, f_2], f_8 = f_7^2 = [f_7, f_1] = [f_7, f_2] = [f_4, f_3], f_5^2 = f_7 f_8$. This is the group number 26369 in the GAP library of groups of order 256 and is of nilpotency class 4. It can be seen that $Z(G) = \langle f_6, f_8 \rangle$ is non-cyclic, $Z_2(G) = \langle f_3, f_4, f_6, f_7, f_8 \rangle$ is non-abelian, and $\Phi(G) = \langle f_5, f_6, f_7, f_8 \rangle$. Observe that $z^2 \in Z(G)$ for all $z \in Z_2(G)$. Therefore,

$$W(G) = \{z \in Z_2(G) \mid z^2 \in Z(G)\} = Z_2(G),$$

and thus $W(G)$ is non-abelian. There are 24576 automorphisms of this group. One of these automorphisms is α, where

$$\alpha(f_2 f_4 f_5 f_6 f_7 f_8) = f_2 f_4 f_5 f_7 f_8, \ \alpha(f_1 f_3 f_4 f_5 f_6 f_8) = f_1 f_3 f_4 f_5 f_6 f_8$$
$$\alpha(f_2 f_5 f_6 f_8) = f_2 f_5 f_8, \ \alpha(f_2 f_3 f_5 f_7) = f_2 f_3 f_5 f_6 f_7.$$

Now, by using the relators of G, we have

$$\alpha(f_i) = f_i \text{ for all } i \in \{1, 3, 4, 5, 6, 7, 8\} \text{ and } \alpha(f_2) = f_2 f_6.$$

It is easy to see that α is a non-inner automorphism of order 2 which fixes $\Phi(G)$ element-wise.

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