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SHORT COMMUNICATIONS

On Commuting Automorphisms of Some Finite p-Groups*

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Abstract—Let G be a group and $\mathrm{Aut}(G)$ be its automorphism group. An automorphism α of G is called a *commuting automorphism* if $\alpha(x)x = x\alpha(x)$ for all $x \in G$. The set of all commuting automorphisms of G is denoted by $A(G)$. We find some cases when $A(G)$ is a subgroup of $\mathrm{Aut}(G)$.

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1. INTRODUCTION

Let G be a group. An automorphism α of G is called a *commuting automorphism* of G if for each $x \in G$, $\alpha(x)$ commutes with x. The set of all commuting automorphisms of G is denoted by $A(G)$. It is well known that $A(G)$ does not necessarily form a subgroup of $\mathrm{Aut}(G)$, but it has a number of interesting properties (see [1]). In 1984, Herstein [2] proposed the following problem: If G is a simple non-Abelian group, then prove that $A(G) = 1$? In 1986, giving an answer to this problem, Laffey [3] proved that if G has no non-trivial Abelian normal subgroups, then $A(G) = 1$. Pettet (see [3]), in his personal communication, observed that if $Z(G) = 1$ and $G' = G$, then $A(G) = 1$. A group G is called an *$A(G)$-group* if the set $A(G)$ forms a subgroup of $\mathrm{Aut}(G)$.

Let $\alpha, \beta \in A(G)$ and $x \in G$. Then, by [1, Lemma 2.2], $\alpha(x) = xc_1$ and $\beta(x) = xc_2$ for some $c_1, c_2 \in C_G(G')$. Therefore, if $C_G(G')$ is Abelian, then

$$[\alpha(x), \beta(x)] = [x^{-1}\alpha(x), \beta(x)] = [x^{-1}\alpha(x), x^{-1}\beta(x)] = 1$$

because x^{-1} commutes with both $\alpha(x)$ and $\beta(x)$. Therefore G is an $A(G)$-group by [1, Lemma 2.4(vi)]. Observe that $Z_2(G) \leq C_G(G')$. Rai [4, Lemma 3.2] proved that if G is a finite p-group, where p is an odd prime, such that $Z_2(G)$ is Abelian, then G is an $A(G)$-group. This raises the obvious question: Is G an $A(G)$-group if $C_G(\Phi(G))$ is Abelian? In Sec. 2, we prove the following result.

Theorem 1. *Let G be a finite non-Abelian p-group such that $C_G(\Phi(G))$ is cyclic. Then G is an $A(G)$-group.*

For $x \in G$, let $[x, G]$ denote the set $\{[x, g] \mid g \in G\}$. Let G be a finite p-group and N be a non-trivial normal subgroup of G. The pair (G, N) is called a *Camina pair* if $N \subseteq [x, G]$ for all $x \in G - N$. A finite p-group G is said to be *Frattinian* if $Z(M) \neq Z(G)$ for all maximal subgroups M of G. A Frattinian p-group G satisfying $C_G(Z(\Phi(G))) = \Phi(G)$ is said to be *strongly Frattinian*. In 2013, Vosooghpour and Akhavan-Malayeri [5] showed that for each $n \geq 5$, there exists a non-$A(G)$ p-group of order p^n. They, in fact, proved that if G is an extra-special p-group of order $\geq p^5$, then G is not an $A(G)$-group. Observe that if G is a finite extra-special p-group, then G is Frattinian and $(G, Z(G))$ is a Camina pair. The converse is true if $cl(G) = 2$ (see Prop. 3.2). The following example shows that the converse is false if $cl(G) \geq 3$.

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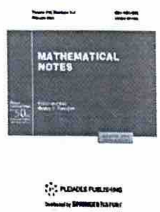

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